The First Fundamental Form

Outline

1. **The First Fundamental Form**

   Let S be a regular surface, and let $\vec{X}\colon U \to S$ be a regular parameterization of an open subset of S. The **first fundamental form** of S with respect to $\vec{X}$ is the matrix

   $$g \;=\; (d\vec{X})^T(d\vec{X})$$

   Since $d\vec{X}$ depends on u and v, the matrix g also depends on u and v. The entries of g are as follows:

   $$g \;=\; \begin{bmatrix} \vec{X}_u \cdot \vec{X}_u & \vec{X}_u \cdot \vec{X}_v \\ \vec{X}_v \cdot \vec{X}_u & \vec{X}_v \cdot \vec{X}_v \end{bmatrix}.$$

2. **Dot Products and Lengths**

   We can use the first fundamental form to take the dot product of tangent vectors to the surface. If $\vec{a}_1$ and $\vec{a}_2$ are vectors at the same point in the uv-plane, and $\vec{t}_1 = d\vec{X}\,\vec{a}_1$ and $\vec{t}_2 = d\vec{X}\,\vec{a}_2$ are the corresponding tangent vectors to the surface, then

   $$\vec{t}_1 \cdot \vec{t}_2 \;=\; (\vec{a}_1)^T g\, \vec{a}_2.$$

   We can also use g to find lengths of tangent vectors to the surface. If $\vec{a}$ is a vector based at a point in the uv-plane and $\vec{t} = d\vec{X}\,\vec{a}$ is the corresponding tangent vector to the surface, then

   $$\|\vec{t}\,\| \;=\; \sqrt{\vec{a}^{\,T} g\, \vec{a}\,}.$$

3. **Lengths of Curves**

   If $\vec{x}(t)$ (for $a \leq t \leq b$) is a curve in the uv-plane, and $\vec{y}(t) = \vec{X}\big(\vec{x}(t)\big)$ is the corresponding curve on the surface, then the length of $\vec{y}$ is

   $$\int_a^b \sqrt{\vec{x}\,'(t)^T g\big(\vec{x}(t)\big)\, \vec{x}\,'(t)}\, dt.$$

   Stated differently, the length of a curve on the surface is $\int ds$, where

   $$ds \;=\; \sqrt{\begin{bmatrix} du & dv \end{bmatrix} \; g \; \begin{bmatrix} du \\ dv \end{bmatrix}}.$$

   This quantity ds is known as the **length element** (or **line element**). In can also be written in terms of the entries of the matrix g:

   $$\text{If} \quad g \;=\; \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad \text{then} \qquad ds \;=\; \sqrt{a\,du^2 + 2b\,du\,dv + c\,dv^2}.$$

4. **Areas of Regions**

It turns out that

$$\|\vec{X}_u \times \vec{X}_v\| \;=\; \sqrt{\det g}.$$

Thus, if R is any region in the uv-plane, and $\vec{X}(R)$ is the corresponding region on the surface, then the area of $\vec{X}(R)$ is

$$\iint_R \sqrt{\det g(u,v)}\, du\, dv.$$

5. **Types of Parameterizations**

We can use $g(u,v)$ to determine the type of a parameterization:

- $\vec{X}$ is equiareal if and only if $\det g(u,v) = 1$ for all u and v.
- $\vec{X}$ is conformal if and only if $g(u,v) = \lambda(u,v)\, I$ for some scalar-valued function $\lambda(u,v)$, where I denotes the 2×2 identity matrix.
- $\vec{X}$ is isometric if and only if $g(u,v) = I$ for all u and v.